

UNITEXT – La Matematica per il 3+2

Volume 85

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Mathematical Analysis II

Second Edition



Springer

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UNITEXT – La Matematica per il 3+2

ISSN 2038-5722

ISSN 2038-5757 (electronic)

ISBN 978-3-319-12756-9

ISBN 978-3-319-12757-6 (eBook)

DOI 10.1007/978-3-319-12757-6

Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014952083

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Cover Design: Simona Colombo, Giochi di Grafica, Milano, Italy

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To Arianna, Susanna and Chiara, Alessandro, Cecilia

Preface

The purpose of this textbook is to present an array of topics that are found in the syllabus of the typical second lecture course in Calculus, as offered in many universities. Conceptually, it follows our previous book *Mathematical Analysis I*, published by Springer, which will be referred to throughout as Vol. I.

While the subject matter known as ‘Calculus 1’ concerns real functions of real variables, and as such is more or less standard, the choices for a course on ‘Calculus 2’ can vary a lot, and even the way the topics can be taught is not so rigid. Due to this larger flexibility we tried to cover a wide range of subjects, reflected in the fact that the amount of content gathered here may not be comparable to the number of credits conferred to a second Calculus course by the current programme specifications. The reminders disseminated in the text render the sections more independent from one another, allowing the reader to jump back and forth, and thus enhancing the book’s versatility.

The succession of chapters is what we believe to be the most natural. With the first three chapters we conclude the study of one-variable functions, begun in Vol. I, by discussing sequences and series of functions, including power series and Fourier series. Then we pass to examine multivariable and vector-valued functions, investigating continuity properties and developing the corresponding integral and differential calculus (over open measurable sets of $\mathbb{R}^n$ first, then on curves and surfaces). In the final part of the book we apply some of the theory learnt to the study of systems of ordinary differential equations.

Continuing along the same strand of thought of Vol. I, we wanted the presentation to be as clear and comprehensible as possible. Every page of the book concentrates on very few essential notions, most of the time just one, in order to keep the reader focused. For theorems’ statements, we chose the form that hastens an immediate understanding and guarantees readability at the same time. Hence, they are as a rule followed by several examples and pictures; the same is true for the techniques of computation.

The large number of exercises, gathered according to the main topics at the end of each chapter, should help the student test his improvements. We provide the solution to all exercises, and very often the procedure for solving is outlined.

Some graphical conventions are adopted: definitions are displayed over grey backgrounds, while statements appear on blue; examples are marked with a blue vertical bar at the side; exercises with solutions are boxed (e.g., 12.).

This second edition is enriched by two appendices, devoted to differential and integral calculus, respectively. Therein, the interested reader may find the rigorous explanation of many results that are merely stated without proof in the previous chapters, together with useful additional material. We completely omitted the proofs whose technical aspects prevail over the fundamental notions and ideas. These may be found in other, more detailed, texts, some of which are explicitly suggested to deepen relevant topics.

All figures were created with MATLABTM and edited using the freely-available package `psfrag`.

This volume originates from a textbook written in Italian, itself an expanded version of the lecture courses on Calculus we have taught over the years at the Politecnico di Torino. We owe much to many authors who wrote books on the subject: A. Bacciotti and F. Ricci, C. Pagani and S. Salsa, G. Gilardi to name a few. We have also found enduring inspiration in the Anglo-Saxon-flavoured books by T. Apostol and J. Stewart.

Special thanks are due to Dr. Simon Chiossi, for the careful and effective work of translation.

Finally, we wish to thank Francesca Bonadei – Executive Editor, Mathematics and Statistics, Springer Italia – for her encouragement and support in the preparation of this textbook.

Torino, August 2014

Claudio Canuto, Anita Tabacco

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